with

$$A_0 = \begin{bmatrix} -1.63 & 1.0 & 0.0 \\ -8.69 & -1.94 & 0.0 \\ 0.0 & 1.0 & 0.0 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0.09 \\ -8.09 \\ 0.0 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The following cost function has been considered:

$$C = \int_0^{+\infty} (\alpha^2 + q^2 + r\delta^2)\,\mathrm{d}t$$

So we have

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad W = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $0 \leq a \leq 1.25$.

Solving the modified Riccati equation we get

$$\text{for} \quad \lambda_{\max} = 0 \qquad S(0) = \begin{bmatrix} 0.304 & 0.0 & 0.0 \\ 0.0 & 0.0362 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\text{for} \quad \lambda_{\max} = 0.2 \qquad S(0) = \begin{bmatrix} 0.450 & 0.0 & 0.0 \\ 0.0 & 0.0453 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$\text{for} \quad \lambda_{\max} = 0.6 \qquad S(0) = \begin{bmatrix} 0.909 & 0.01 & 0.0 \\ 0.01 & 0.0835 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

It has been found also that only five other reference points are necessary to cover the whole operations domain when the guaranteed performance level is chosen 50% above the reference level.

VI. Conclusion

A new technique derived from the guaranteed cost control method of Chang and Peng has been developed for the closed-loop control of systems with varying parameters. This technique allows the definition of guaranteed performance and stability regions around reference points. An heuristic approach is then available for the definition of a set of reference points that conveniently cover the whole operations domain. This approach seems particularly promising for aerospace applications.

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Optimal Rocket Steering in Terms of Angular Velocity of the Primer Vector

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Introduction

LAUNCHING a rocket from the Earth's surface into orbit requires a level of performance significantly above that required to achieve orbital velocity alone, about 8 km/s. It is found that the required performance as measured by the so-called ideal-velocity gain is between 10 and 11 km/s. The shortfall is due to several losses: 1) atmospheric, 2) steering, and 3) gravity.

The first results from atmospheric drag and reduced engine efficiency (due to the exhaust having to push aside the atmosphere). Steering losses result from (possibly) conflicting demands between altitude and path-angle requirements and a need to escape the atmosphere. This phenomenon forces the initial thrust to be directed more upward than otherwise desirable. Gravity losses result from the need to counter gravity via propulsion once the support of the launch platform is lost.

This difference in ideal-velocity gain between 11 km/s vs 8 km/s is quite significant and results in a payload loss of about 50%. (The difference goes to additional propellant.) Since with current booster technology it costs many hundreds of dollars to place one kilogram into orbit, any savings in required velocity gain are welcome indeed! Such savings are realized by optimal shaping of the launch trajectory.

These basic principles were recognized as far back as Tsiolkovsky,[1] who performed elementary calculations for inclined vs vertical ascents and showed that significant savings accrue from the former. His mathematical techniques, while robust, were not sufficient to arrive at a precision optimal trajectory. Credit goes to Goddard[2,3] for first recognizing the importance of the calculus-of-variations for trajectory shaping. His analysis was directed toward vertical ascents of sounding rockets, consistent with his limited stated goal of reaching "extreme altitudes." To Oberth[4,5] we owe the term "synergistic trajectory" to denote a trajectory balanced with regard to the various performance losses—in other words, an optimal trajectory. Although he developed his ideas in some detail, his methods are not based on the calculus-of-variations.

Preliminaries

The detailed rigorous solution to the optimal rocket problem was worked out in the 1950s by Lawden[6] and others.[3,7–10]

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Lawden's solution remains the most elegant:

$$\ddot{p} = (p \cdot \nabla)g \tag{1}$$

where p is Lawden's primer vector and is in the direction thrust should be applied, ∇ is the gradient operator, and g is the local gravity field. The extreme simplicity of this equation is deceptive, however. Because g is a function of the position r, the terms involving the derivatives of $g(r)$ are generally not known until the trajectory has been established—in other words, until the problem has been solved. In terms of complexity it appears that the solution to Lawden's equation is somewhere between the two- and three-body problems—or just out of reach of general closed-form solution.

The most important special case admitting a closed-form solution occurs when the gravity field may be treated as uniform. In this event the right-hand side of Eq. (1) vanishes and the general solution for p becomes

$$p = at + b \tag{2}$$

where a and b are arbitrary constant vectors. Note that this forces the primer vector to lie in a fixed plane (for a given case). We may take this plane to be the x-y plane, with y parallel to a. Then if χ is the angle the primer vector makes with the x axis from Eq. (2), we obtain[6,11]

$$\tan \chi = a_1 t + b_1 \tag{3}$$

where a_1 and b_1 are now scalars (and different from previous usage). By differentiating Eq. (3) [or from Eq. (2)] we can obtain the expression form the angular velocity of the primer vector:

$$\eta \overset{d}{=} \dot{\chi} = \eta_m / [1 + \eta_m^2 (t - t_m)^2] \tag{4}$$

where we have used η to represent the pitch rate $\dot{\chi}$, η_m the maximal value of the pitch rate, and t_m the time at which the maximal value occurs. Note that t_m need not fall within the thrusting interval, thus the maximal value may not actually be attained, although it usually is. Observe that if $\eta_m(t - t_m) \ll 1$, the pitch rate will be very nearly uniform (constant) over the thrust period.

Parametric Guidance

Lawden's equation is difficult to solve in the general case, especially in real time. "Parameterized guidance" attempts to circumvent this by restricting the form of the guidance law to certain prespecified functionals. Free parameters within the guidance functional allow enough trajectory shaping to enable one to achieve the desired cutoff conditions—and also give the topic its name. A variety of functional forms have been investigated. To be effective, the functional must be simple, capable of approximating well the true optimal solution, and free of numerical singularities or "surprises."

The constant gravity solution, besides providing valuable insight into the nature of the general solution, is a natural candidate for parameterized guidance. The development of the linear-tangent law (constant gravity) into a highly successful mode of parameterized guidance is primarily associated with Perkins.[11] References 15 and 16 show that the linear-tangent law is still very much alive and well.

A number of investigators have noted that actual trajectories (from numerical calculations) show a pitch rate apparently more uniform than would be expected based on the linear-tangent law. Accordingly, Smith[13] and coworkers have suggested a linear-angle, rather than linear-tangent, law. Although computationally successful, it has proven difficult to justify this particular functional form, other than as an approximation of the linear-tangent law. Moreover, since in some successful applications the constant-gravity assumption is seriously strained, this mode of analysis is not altogether convincing, as shown in Pfeiffer.[12]

The following analysis suggests that under some conditions a "straightening out" of the pitch profile does indeed occur. It also suggests a more rigorous approach for parametric guidance.

Lawden's Equation in a Rotating Frame

If Eq. (1) is expressed in a rotating coordinate frame, having arbitrary angular velocity η, there results

$$\ddot{p} + 2\eta \times \dot{p} + \dot{\eta} \times p - |\eta|^2 p + (\eta \cdot p)\eta = (p \cdot \nabla)g \tag{5}$$

All vectors are to be represented in the rotating system. The angular velocity vector, η, is perfectly arbitrary at this point, and need have no relation to the primer vector. (The last term on the left-hand side of Eq. (5) has been retained for complete generality. Actually even if the trajectory is not coplanar this term may be dropped without loss of generality. For on resolving η into vectors parallel and orthogonal to p one finds that the parallel part has no effect, and so may be dropped. The indicated term then also vanishes.) If now we take η to be the angular velocity of p, then we are justified in assigning a fixed direction to p in the rotating system. In particular we can take the x axis to always lie along p. Suppose in addition the trajectory is confined to a plane; then the primer vector will also be confined to this plane, and the vector η may be taken normal to this plane. Letting x-y denote the trajectory plane, with p along x, η will now always lie along z. Thus η may be treated as a scalar, and the resulting equations for p, in terms of its two components, are given by

$$\ddot{p}_x - \eta^2 p_x = g_{xx} p_x \tag{6a}$$

$$2\eta \dot{p}_x + \dot{\eta} p_x = g_{yx} p_x \tag{6b}$$

where

$$g_{xx} \overset{d}{=} \frac{\partial g_x}{\partial x} \quad \text{and} \quad g_{yx} \overset{d}{=} \frac{\partial g_y}{\partial x}$$

In deriving these equations explicit use was made of the fact that the y and z components of p vanish identically. We can therefore dispense with the subscript x on p, because in our case $p_x = |p| = p$.

Accordingly, we can rewrite Eqs. (6) as

$$\ddot{p} = (g_{xx} + \eta^2)p \tag{7a}$$

$$\dot{p} = \frac{(g_{yx} - \dot{\eta})p}{2\eta} \tag{7b}$$

Elimination of the Magnitude and the Equation of the Pitch-Rate

The next step is to eliminate p from Eqs. (7) and arrive at an equation involving only the pitch-rate η:

$$2\eta\ddot{\eta} - 3\dot{\eta}^2 + 4\eta^4 = -4\eta^2 g_{xx} - 4g_{yx}\dot{\eta} + 2\frac{\mathrm{d}g_{yx}}{\mathrm{d}t}\eta + g_{yx}^2 \tag{8}$$

This is a second-order, fourth-degree equation in η. Note that all "forcing" terms, those involving gravity, have been placed on the right-hand side of Eq. (8). For a constant gravity field these terms drop out, and it is easy to show that the reduced equation is then satisfied by Eq. (4). (We shortly derive a more general expression.) In passing we note that, from Eq. (7b), the magnitude of the primer vector is given by

$$p = k \cdot \eta^{-0.5} \exp\left[\int g_{yx}/(2\eta)\, \mathrm{d}t \right] \tag{9}$$

where k is a constant. In the case where g is constant, this equation can easily be shown to be consistent with Eqs. (2) and (4). Equation (9) suggests that the primer vector is probably tilting fastest when its magnitude is minimal; for the linear-tangent case this can be confirmed directly.

Gravity-Gradients for the Inverse-Square Force Field

Before proceeding with the main analysis, we need explicit expressions for the gravity-gradient terms g_{xx} and g_{xy} for an inverse-square (coulomb) force field. These gravity-gradient terms will be expressed in terms of the pitch angle above the local horizontal ψ and the Schuler angular velocity ω. The latter, it will be recalled, is the angular velocity of a circular orbit and is given by

$$\omega = \sqrt{\mu/R^3} \tag{10}$$

In terms of ψ and ω we have

$$g_{xx} = \omega^2(1/2 - 3/2 \cos 2\psi) \tag{11a}$$

$$g_{yx} = 3/2\ \omega^2 \sin 2\psi \tag{11b}$$

We also need the time derivative of g_{yx}. This is given by

$$\frac{\mathrm{d}g_{yx}}{\mathrm{d}t} = \omega^2[3 \cos(2\psi)\, \dot{\psi} - (9/2) \sin(2\psi)\, \dot{R}/R] \tag{12}$$

In interpreting the equation, it must be kept in mind that x is along the primer vector and y orthogonal to it. In general these are not aligned with the horizontal and vertical. Note that the angle ψ appears in duplicate. Also, as g is (usually) the gradient of a potential, we must have $g_{xy} = g_{yx}$, etc. Finally, by Laplace's equation, $g_{xx} + g_{yy} + g_{zz} = 0$. These relationships are handy for checking results. Another useful fact is that for low-Earth orbits, $\dot{\psi}$ is about four deg per minute.

A diagram of the coordinate system is shown in Fig. 1. In keeping with the tradition, the trajectory is shown going from west to east (left to right). This means that θ and ψ will be decreasing, so that $\dot{\psi}$ and η will be negative.

Integration of the Pitch-Rate Equation

We now show that Eq. (8) can be solved by quadratures, providing that the right-hand side can be written as a function of η. To this effect rewrite Eq. (8) as

$$2\eta\ddot{\eta} - 3\dot{\eta}^2 + 4\eta^4 = q(\eta) \tag{13}$$

Next define

$$Q \stackrel{d}{=} (1/4\eta) \int (q(\eta)/\eta^4)\mathrm{d}\eta \tag{14}$$

then we have

$$\int \frac{-\mathrm{d}\eta}{2\eta^2\sqrt{k/4\eta - 1 + Q(\eta)}} = t - t_0 \tag{15}$$

where k and t_0 are arbitrary constants of integration. Upon performing the integration and inverting the result we obtain the required $\eta(t)$. Of course the rub in all of this is that we do not know q as a function of η, but rather as a complicated expression involving many parameters, given by the right-hand side of Eq. (8). Nevertheless, we can propose various functional forms for $q(\eta)$ and investigate the implications for $\eta(t)$. A second, more systematic approach is to introduce a number of arbitrary constants (parameters) into the functional form of q, so that we may write $q(\eta, k_1, k_2 \ldots)$, then choose k so as to be as consistent as possible with Eq. (8). For example, if Q is limited to a fourth-degree polynomial in $1/\eta$, then the quadrature of Eq. (15) can be always effected in terms of Jacobian elliptic functions. From Eq. (14) we see that this implies q is a fourth-degree polynomial in η, but missing the cubic term. Thus for a rather substantial family of $q(\eta)$, closed-form solutions can be achieved. To develop this fully is rather complicated, so we will limit ourselves to the following important special case.

Special Case

Suppose that $q(\eta) = 4k_0\eta^4$, where k_0 is an arbitrary constant. Then it follows that $Q(\eta) = k_0$, and the solution of η is given by

$$\eta(t) = \eta_m/[1 + (1 - k_0)\eta_m^2\ (t - t_m)^2] \tag{16}$$

where, as previously, η_m denotes the maximal pitch rate (which, recall, may not be achieved because it falls outside the thrusting period). Functionally this expression is almost identical to Eq. (4) except for the added $(1 - k_0)$ factor. When there is no gravity-gradient, k_0 vanishes and Eq. (16) becomes identical to Eq. (4), as expected. Notice that if k_0 is positive (and less than unity) the curve of Eq. (16) becomes flatter than in the gravity-free case. The limiting case is when $k_0 = 1$, in which case $\eta(t)$ becomes a constant. In this event all the derivatives of η must vanish, and the equation η must satisfy can be obtained either from $4k_0\eta^4 = q(\eta)$ or directly from Eq. (8):

$$4\eta^4 = -4\eta^2\, g_{xx} + 2\frac{\mathrm{d}g_{yx}}{\mathrm{d}t}\eta + g_{yx}^2 \tag{17}$$

The expressions for the gravity-gradient were given in Eqs. (10–12). These can be somewhat simplified by noting that in the early phase of the trajectory we will have $\dot{R}/R \ll \omega$ and $\dot{\theta} \ll \omega$, so that $\dot{\psi} = \eta$. With these substitutions Eq. (17) becomes

$$4\eta^4 = -4\eta^2\omega^2\ [1/2 - (3/2)\cos 2\psi] + 6\eta^2\omega^2\cos(2\psi) + (9/4)\omega^4\sin^2 2\psi \tag{18}$$

In interpreting this equation it must be recollected that initially a rocket needs to climb steeply to clear the ground and escape the atmosphere; if it were not for these demands the optimal path would be much more horizontal. If we select a point on the trajectory right after the denser part of the

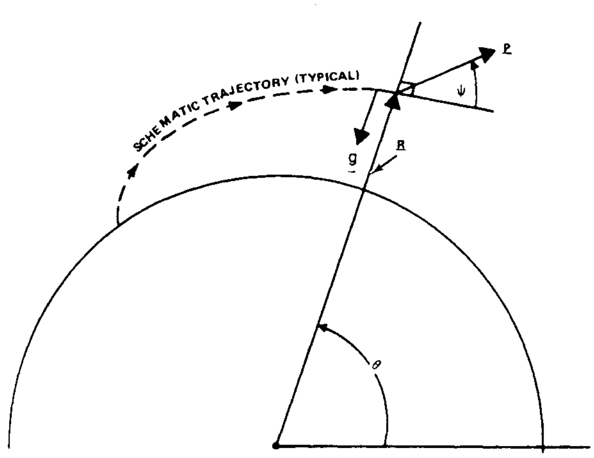

Fig. 1 Trajectory and guidance geometry.

atmosphere has been exited but before a significant part of orbital velocity has been achieved, values of ψ in the range 30–45 deg are appropriate, rather than the launch value of 90 deg. These then are the values that should be used in Eq. (18).

In a similar manner it is possible to treat the case at the instant orbital conditions are achieved. The principal deviation from the previous is that now we must have $\dot{\psi} = \eta - \dot{\theta} = \eta + \omega$. The solution for this is shown in Ref. 14. For η to be constant, the same value must apply to both the launch and orbital end conditions; graphical analysis then shows that only ηs in the narrow range of about $-0.7\ \omega$ to $-1.0\ \omega$ are admissible. From the equation

$$\Delta\psi + \Delta\theta = \eta T \tag{19}$$

we can estimate the thrusting time T. (The $\Delta\theta$ is the central angle traversed during the maneuver and is approximately equal to $\omega T/2$. The $\Delta\psi$ is the difference between the final and initial pitch angles in the local frame, and is equal to about 30 deg when $\eta = -\omega$. Both Δ are negative, but then so is η according to our convention.) The estimate produced by Eq. (19) is about 900 s, which is reasonable but on the high side.

Conclusion

Actual launch vehicles have a higher acceleration profile and are able to achieve orbit in about 600 s. Consequently their pitch-rate magnitude is more in the range 1.5 to 2 ω, and falls out of the previous range. Thus, while exactly linear pitch profiles probably do not apply, the above analysis suggests why a nearly constant pitch rate is tenable. (See also Ref. 14.)

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Invariant Set Analysis of the Hub-Appendage Problem

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Introduction

IN the recent literature, an asymptotic stability theorem[1] for autonomous and periodic nonautonomous systems was used to prove the global asymptotic stability of the mass-spring-damper system and the damped Mathieu system. For such systems, the application of LaSalle's invariant set theorem[3] has been the conventional approach adopted to prove the global asymptotic stability. When the derivative of the Lyapunov function[2] vanishes, LaSalle's theorem[3] requires us to show that the maximum invariant set of the system consists only of the equilibrium point at its entry. Although it is always simple to identify the set of points Q where the derivative of the Lyapunov function vanishes, the maximum invariant set $I \subset Q$ is not always easy to identify. The main challenge of LaSalle's theorem[3] is therefore to sort out the maximum invariant set. For a distributed parameter system the dynamics are described by a hybrid set of ordinary and partial differential equations. For such a system, the sorting out of the maximum invariant set is not a trivial task. In such a situation it is useful to apply the theorem in Ref. 1 so as to comment on the asymptotic stability of the system.

The distributed parameter system consisting of a rigid hub with one or more cantilevered flexible appendages has appeared in the technical literature quite frequently (see Refs. 4, 5, 6, and 7). The system described in Fig. 1 consists of four appendages that are identical uniform beams conforming to the Euler-Bernoulli assumptions. Each beam cantilevered rigidly to the hub is assumed to have a tip mass. The motion of the system is confined to the horizontal plane and the control torque is generated by a single-reaction wheel actuator. Under the assumption that the system undergoes antisymmetric motion with deformation in unison (see Fig. 2), a class of rest-to-rest maneuvers was considered in Ref. 4. For the particular Lyapunov function considered, the best choice of the control input only guaranteed the negative semidefiniteness of the derivative of the Lyapunov function. To conclude the global asymptotic stability using LaSalle's theorem, it would be necessary to formally prove that the maximum invariant set consists only of the equilibrium point. The global asymptotic stability of the system was claimed in Ref. 4 in the absence of this proof.

In this Note we consider the hub-appendage problem[4] with modifications. The modeling and successful control of such a system is expected to provide us with insight into the modeling and control of a general class of distributed parameter systems. Using a Lyapunov function approach and the asymptotic stability theorem in Ref. 1, we prove that global asymptotic stability of the system is guaranteed provided the system undergoes antisymmetric motion with deformation in unison.

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